# Math 210B Lecture 11 Notes

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## 1 Tensor Products

### 1.1 Construction, universal property, and examples

Let $A$ be a ring, let $M$ be a right $A$-module, and let $N$ be a left $A$-module.

**Definition 1.1.** The **tensor product** of $M$ and $N$ over $A$, denoted $M \otimes_A N$, is the quotient of $\mathbb{Z}^{M \times N} = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}(m,n)$ by the $\mathbb{Z}$-submodule generated by

1. $(m+m', n) - (m,n) - (m',n)$
2. $(m, n'+n) - (m,n) - (m,n')$
3. $(ma, n) - (m, an)$

for all $m, m' \in M$, $n, n' \in N$, and $a \in A$. The image of $(m,n)$ in $M \otimes_A N$ is denoted $m \otimes n$ and is called a **simple tensor**.

**Example 1.1.** How do simple tensors work? Let $k \in \mathbb{Z}$.

$$k(m \otimes n) = (m \otimes n) + \cdots + (m \otimes n) = (m + \cdots + m) \otimes = (km) \otimes n = m \otimes (kn).$$

Similarly,

$$(-1)(m \otimes n) = (-m) \otimes n.$$

$$0 \otimes n = 0 = m \otimes 0.$$

**Proposition 1.1** (tensor product universal property)**.** *Let $L$ be an abelian group and $\phi : M \times N \to L$ be such that*

1. *$\phi(m+m', n) = \phi(m,n) + \phi(m',n)$ (left biadditivity)*
2. *$\phi(m, n+n') = \phi(m,n) + \phi(m,n')$ (right biadditivity)*
3. *$\phi(ma, n) = \phi(m, an)$ ($A$-balanced).*

*There exists a unique homomorphism $\Phi : M \otimes_A N \to L$ such that $\Phi(m \otimes n) = \phi(m, n)$ for all $m \in M$ and $n \in N$.*

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[d] & L \\ M \otimes_A N \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

*Proof.* $M \otimes_A N = \mathbb{Z}^{M \times N}/I$ for the ideal generated by the relations. $\mathbb{Z}^{M \times N}$ is free over $\mathbb{Z}$, so there exists a unique $\varphi : \mathbb{Z}^{M \otimes N} \to L$ given by $\varphi((m, n)) = \phi(m, n)$. We get

$$\begin{tikzcd} \mathbb{Z}^{M \times N} \arrow[r] \arrow[d] & L \\ M \otimes_A N \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

whrer the map $\mathbb{Z}^{M \otimes N} \to M \otimes_A N$ is surjective. This uniquely determined $\Phi$ if it exists; i.e. $\Phi(I) = 0$. We can verify, for example, that

$$\varphi((m + m', n) - (m, n) - (m, n)) = \phi(m + m; , n) - \phi(m, n) - \phi(m', n) = 0. \qquad \square$$

Here is a special case. Let $A$ be an $R$-algebra, where $R$ is commutative. Let $\psi : R \to Z(A)$, the center of $A$. $M$ is an $R$-$A$ bimodule, where $rm = mr$. Recall that an $A$-$B$ bimodule is a left $A$-module and a right $B$ module such that $(am)b = a(mb)$ fir all $a \in A$, $m \in M$ and $b \in B$. We can define

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn)$$

to give $M \otimes_A N$ an $R$-module structure. Another way to do this would be to deinfe $M \otimes_A N$ as $R^{M \times N}$, quotiented by the $R$-submodule generated by the same relations, plus the relation $r(m, n) - (rm, n)$.

What is the universal property saying?

$$\mathrm{Hom}_{R-\mathrm{mod}}(M \otimes_R N, L) \cong \mathrm{Hom}(M \times N, L),$$

where the right side is homomorphisms that are $R$-bilinear and $A$-balanced.

**Example 1.2.** Let $K$ be a field. Then $K^m \otimes_K K^n$ is an $mn$-dimensional $K$ vector space, generated by $e_i \otimes e_j$, where $\{e_i\}$ and $\{e_j\}$ form a basis for $K^m$ and $K^n$, respectively:

$$K^m \otimes K^n = \left( \bigoplus_{i=1}^{m} K \right) \otimes K^n \cong \bigotimes_{i=1}^{m} (K \otimes K^n) \cong \bigoplus_{i=1}^{m} K^n \cong K^{mn}.$$

**Example 1.3.** $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$. We have the biadditive, $\mathbb{Z}$-balanced map $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ sending $(a,b) \mapsto ab$, so there exists a unique map $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ sending $a \otimes b \mapsto ab$. This is surjective. Let $a, b \in \mathbb{Z}$. Then $m(a \otimes b) = ma \otimes b = 0$, and $n(a \otimes b) = a \otimes nb = 0$. Also, $a \otimes b = ab(1 \otimes 1)$, which means that this group is cyclic by has order dividing $m$ and dividing $n$. So the map is injective.

**Example 1.4.** $A \otimes_A N \cong N$ as let $A$-modules.

More generally, let $A, B, C$ be rings, let $A$ be an $A$-$B$ bimodule, and let $N$ be a $B$-$C$ bimodule. Then $M \otimes_B N$ is an $A$-$C$ bimodule.

$$a(m \otimes n) = (am) \otimes n, \qquad m \otimes (nc).$$

## 1.2 Properties of the tensor product

**Proposition 1.2.** *$M \otimes_A \cong N \otimes_{A^{\mathrm{op}}} M$.*

*Proof.* We have the map $(m,n) \mapsto m \otimes n$ which is bilinear and $A$-balanced. It induces a unique map $M \otimes_A N \to N \otimes_{A^{\mathrm{op}}} M$. $\square$

**Proposition 1.3.** *Let $L$ be a right $A$-module, let $M$ be an $A$-$B$ bimodule, and let $N$ be a left $B$-module. Then $(L \otimes_A M) \otimes BN \cong L \otimes_A (M \otimes_B N)$.*

*Proof.* We can verify this using the universal property, as before. Alternatively, we can define the object $L \otimes_A M \otimes_B N$ as we defined the tensor product and show that $(L \otimes_A M) \otimes BN$ and $L \otimes_A (M \otimes_B N)$ are isomorphic to it. $\square$

**Proposition 1.4.** *$(\bigoplus_{i \in I} M_i) \otimes_A N \cong \bigoplus_{i \in I} (M_i \otimes AN)$.*

**Proposition 1.5.** *Let $M$ be a left $A$-module, and let $I \subseteq A$ be a 2-sided ideal. Then $A/IA \otimes_A M \cong M/IM$ as $A$-modules.*

*Proof.* Define a map $\phi : A/IA \times M \to M/IM$ such that $\phi(\overline{a}, m) = am + IM$. This is well-defined because if $b \in I$, then $\phi(b, m) = bm + IM = 0$. This satisfies the properties we need, so there exists a homomorphism $\Phi : A/I \otimes_A M \to M/IM$ of $A$-modules. This homomorphism is surjective. We can define an inverse $M/IM \to A/IA \otimes_A M$ sending $m + IM \mapsto 1 \otimes m$; this is well-defined because for $b_i \in I$ and $m_i \in M$,

$$\sum b_i m_i \mapsto 1 \otimes \sum b_i m_i = \sum (1 \otimes b_i m_i) = \sum \underbrace{(b_i \otimes m_i)}_{=0} = 0.$$

Check that this is the inverse of $\Phi$. $\square$

We can also take tensor products of $R$-algebras $A$ and $B$ to get and $R$-algebra $A \otimes_R B$, where $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$.